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Approximation of System Reliabilities Using a Shooting Monte Carlo Approach

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A new Monte Carlo based method is presented for estimating the system reliability of constraint bounded design spaces. The new method is first developed in its general form, which is applicable to any joint probability distribution. It is also demonstrated that this method can be used to concurrently estimate derivatives of the system reliability (e.g., for use in gradient-based numerical optimization). The method is then specialized to the particular case of an *n*-dimensional Gaussian (normal) distribution, which allows for a simpler form. An important property of probability distributions, the dead band effect, is also presented, and it is shown that this effect has important ramifications for the application of the shooting Monte Carlo approach. Finally, numerical values are presented that demonstrate improved efficiencies of up to orders of magnitude relative to the conventional Monte Carlo approach.

Introduction

REAL engineering parameters, such as material properties or applied loads, are rarely known with absolute certainty. For example, when a structural material is qualified for its yield modulus, a large number of nominally identical test coupons will be made and then each one tested individually. The result will not be a single number but rather some probability distribution for the parameter in question. However, because of the difficulties inherent in treating multidimensional probability spaces, traditional methods of engineering analysis have ignored these probabilistic effects. In the case of properties such as elastic modulus, the mean value would typically be used, with the statistical deviation simply ignored. For failure critical parameters such as yield modulus, one is forced to choose some value Y such that x% of the specimens are expected to survive up to at least Y. For obvious reasons this value is typically chosen very conservatively. As the design model is built up, a great number of these parameters may appear, leading to a design whose reliability is typically much higher than required, but in any case, is simply unknown. In an increasing number of modern engineering design problems, it is desired to calculate, or at least estimate, these reliabilities explicitly, either to ensure that the design does, in fact, meet some given system level reliability criteria, or, more commonly, to trade the excess reliability in the design for some desired result such as reduced weight or cost. It is because of the latter application that reliability-based methods, which explicitly include these probabilistic effects for at least some critical parameters, are of particular interest within the field of numerical optimization; it is within this context that this paper is set.

Parameter Subsets

Let the complete set of independent parameters for some problem be defined as $\boldsymbol{Q}$. In general some subset $\boldsymbol{x}$ will be considered variable design parameters, i.e., their values will be allowed to vary during the course of the optimization, while the remainder $\boldsymbol{y}$ are considered fixed design parameters. In addition, the parameters $\boldsymbol{Q}$ can be independently divided into two subsets $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a}$ are the probabilistic parameters and the remainder $\boldsymbol{b}$ are considered deterministic parameters. Reliability calculations are dependent on all of the parameters in the $\boldsymbol{a}$ set, whether or not they are also design variables in the $\boldsymbol{x}$ set.

Conventional Monte Carlo Integration/Simulation

Appendix A presents a summary of the conventional Monte Carlo (CMC) approach as applied to the evaluation of reliability for an n-dimensional probability space. Appendix A also derives an explicit formula for the expected number of CMC design point evaluations required for a given level of relative accuracy, and it is shown that this number quickly rises toward infinity as the system reliability rises to 1 ($R_S \longrightarrow 1$).

Gaussian Distribution

Appendix B presents a summary of the Gaussian distributions (also known as normal distributions). These appear quite commonly in the characterization of engineering parameters, and methods have been developed to at least approximate other distributions in this form. In addition, it is shown that Gaussian distributions in multivariable space have some very special properties; the most important is that they can be normalized so as to be spherically symmetric in any n-dimensional probability space. Although both the conventional and the shooting Monte Carlo (SMC) methods are applicable to problems with arbitrary joint probability distributions, the SMC approach will be shown to reduce to a more efficient form for the special case of the n-dimensional Gaussian distribution.

Reliability as a Probability Integral

Consider the simple example of a single probabilistic parameter, such as the length of a steel bar. If we were to order a large number of bars, each with a nominal length L, and then carefully measure the actual length L_k of each of these bars, we would find that in fact none of these bars have a length of exactly L. Instead, the length of each instance of the set of bars will have some variance δL from its nominal value,

$$L_k = L + \delta L_k \tag{1}$$

If we now pick some critical length Z, then the fraction of actual lengths $L_k \leq Z$ is denoted by the cumulative distribution function $\Phi(Z)$, while the fraction of actual lengths $L_k > Z$ is simply $1 - \Phi(Z)$. Any probabilistic parameter can be characterized this way, and although the shape of $\Phi(Z)$ will vary between different parameters it will always rise monotonically from $\Phi(-\infty) = 0$ to

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$\Phi(+\infty) = 1$. Note that for this example the distribution function could be written either in terms of the total lengths L_k, as shown, or in terms of the variances δL_k by simply shifting the value of Z by the nominal length L.

In probability theory it is common to define the distribution of actual values for a probabilistic parameter v in terms of its probability density function $\phi(\delta v)$, which is defined such that

$$P(\delta v < Z) = \Phi(Z) \equiv \int_{-\infty}^{Z} \phi(\delta v)\, \mathrm{d}\delta v \tag{2}$$

where $\phi(\delta v)$ will typically be a function of the nominal value of v, as well as other parameters that control the shape and extent of the distribution of δv. In the literature one more commonly finds the cumulative and density functions written in terms of the total instance values $v + \delta v$, which will often be denoted as simply v. For this paper, however, it is advantageous to maintain the distinction between the nominal value of a parameter and its probabilistic variance from this nominal value.

Consider next the more realistic scenario in which a design is defined in terms of many parameters $\boldsymbol{Q}$, of which some n-dimensional subset $\boldsymbol{a}$ will be considered to have actual values with significant statistical deviation from their nominal values. For some fixed nominal values $\boldsymbol{a}$ one can define the probability that the actual values $\boldsymbol{a} + \delta\boldsymbol{a}$ lie in some region Ω of the n-dimensional probabilistic design space as

$$P(\boldsymbol{a} + \delta\boldsymbol{a} \in \Omega) = \Phi_n(\Omega) \equiv \int_{\Omega} \phi_n(\boldsymbol{a} + \delta\boldsymbol{a})\, \mathrm{d}\Omega \tag{3}$$

Similarly, the success or failure of each instance of the nominal design will be determined by many individual analysis criteria, or constraints, such that if any of the constraints are violated, $g_i(\boldsymbol{Q} + \delta\boldsymbol{Q}_k) > 0$, then the kth design is considered to have failed. All successful instances of the design, therefore, must fall into the feasible region of the design space, for which all of the individual constraints are satisfied as $\boldsymbol{g}(\boldsymbol{Q} + \delta\boldsymbol{Q}_k) \leq 0$. The system reliability R_S can then be written in terms of the joint probability density of the $\boldsymbol{a}$ probabilistic parameters as

$$R_S = \Phi_n(\Omega) = \int_{\Omega} \phi_n(\boldsymbol{a} + \delta\boldsymbol{a})\, \mathrm{d}\Omega \tag{4}$$

where Ω is now defined to be the projection of the feasible region formed by fixing all of the deterministic parameters $\boldsymbol{b}$ at their nominal values. Unfortunately, this integral can almost never be evaluated explicitly, either analytically or numerically. The individual constraints are generally not independent, because any given point in the infeasible region may violate only one constraint, or a few constraints, or even all of the constraints simultaneously. Because the correlations between the constraints cannot be determined a priori for any but the simplest constraint types, methods such as the safety index approach, which estimate the reliability of each constraint individually, cannot generally be used to estimate the total system reliability. Although direct numerical integration schemes can be applied to Eq. (4) when the dimensionality n is less than four or five, above that they rapidly break down. The approach most often taken in these cases is to estimate R_S using CMC simulation, which is summarized in Appendix A. Unfortunately the CMC approach has two important flaws.

1) It can be very expensive in terms of number of point design analyses required, particularly as $R_S \longrightarrow 1$.

2) It does not provide the design sensitivity information (e.g., derivatives of R_S with respect to the nominal values $\boldsymbol{Q}$) required to use reliability in gradient-based optimization schemes.

In the next section a new variant of the Monte Carlo approach is presented, which reduces the number of analyses required, while allowing the derivatives of the reliabilities to be estimated simultaneously with the reliabilities themselves.

SMC

Consider some n-dimensional probability space, where each of the n independent probabilistic parameters has a nominal value a_i and some variance δa_i. Each individual parameter has a probability density function $\phi_i(a_i + \delta a_i)$ associated with it, and so the set of parameters $\boldsymbol{a}$ has a joint probability density function of

$$\phi_n(\boldsymbol{a} + \delta\boldsymbol{a}) = \prod_{i=1}^{n} \phi_i(a_i + \delta a_i) \tag{5}$$

Let the feasible region be bounded by the constraint $g \leq 0$, and let each axis of this space be normalized (i.e., nondimensionalized) by scaling with some suitable metric. Then the integral of ϕ_n over this region can be converted to hyperspherical coordinates centered on the nominal value point $\boldsymbol{a}$ as

$$\Phi_n(\boldsymbol{a}, \Omega) = \int_{\Omega} \phi_n(\boldsymbol{a} + \delta\boldsymbol{a})\, \mathrm{d}\Omega = \int_{0}^{1} \eta(\triangleleft)\, \mathrm{d}\triangleleft \tag{6}$$

where $\triangleleft$ represents a generalized n-dimensional angle whose values, ranging from 0 to 1, exactly sweep the hypersphere. The integrand $\eta(\triangleleft)\, \mathrm{d}\triangleleft$ then must represent the integral of ϕ_n over a differential generalized solid angle centered on $\triangleleft$, which implies that $\eta(\triangleleft)$ must represent the cumulative probability Φ_n that would be found if one integrated over the entire hypersphere, and it had a spherically symmetric probability distribution equal to the distribution actually found along the angle $\triangleleft$, from the origin at $\boldsymbol{a}$ to the constraint boundary at $g = 0$ (note that as $\mathrm{d}\triangleleft \longrightarrow 0$ the tangential variation within the differential solid angle becomes negligible):

$$\begin{aligned} \eta(\triangleleft) &= \int_{0}^{r(\triangleleft)} \phi_n(\boldsymbol{a} + \alpha\xi) S_n \alpha^{n-1}\, \mathrm{d}\alpha \\ &= \int_{0}^{r(\triangleleft)} S_n \alpha^{n-1} \left[\prod_{i=1}^{n} \phi_i(a_i + \alpha\xi_i) \right] \mathrm{d}\alpha \end{aligned} \tag{7}$$

where

$\xi(\triangleleft)$ = shooting vector along the generalized angle $\triangleleft$
$r(\triangleleft)$ = positive distance from the origin at $\boldsymbol{a}$ to the boundary of the feasible region along the angle $\triangleleft$, i.e., along the, shooting vector ξ
$S_n \alpha^{n-1}$ = generalized differential volume of a hypersphere; Eq. (B7)

Note that because the total cumulative probability over all space must equal 1, $\eta(\triangleleft)$ will always be finite, monotonically increasing from $\eta = 0$ at $r = 0$ to $\eta \longrightarrow 1$ as $r \longrightarrow \infty$. Figure 1 shows a simple two-dimensional design space, along with three possible shooting vectors.

Monte Carlo Evaluation

Although this transformation has converted the original n-dimensional Cartesian integral into a one-dimensional integral over the generalized angle $\triangleleft$, in general, it is not possible to solve this equation exactly either. However, it can be estimated by taking a Monte Carlo integration over the generalized angle $\triangleleft$. The only theoretical requirement to do this is that the function $\eta(\triangleleft)$ must be single valued in $\triangleleft$, which will always be true; however, to avoid the problem of potentially having to integrate over bands of feasible and infeasible regions (which contribute nothing to the integral), it is convenient to assume that $\boldsymbol{Q}$ is in the feasible region and that

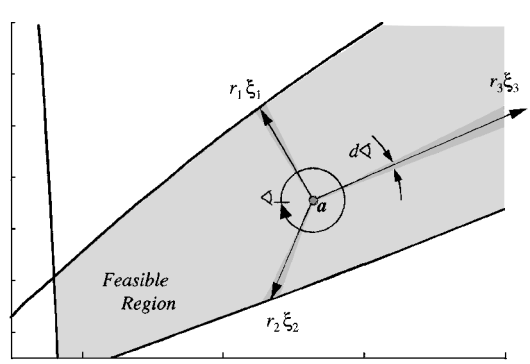

Fig. 1 Each $\eta(\triangleleft)\, \mathrm{d}\triangleleft$ represents a differential solid angle through probability space.

$r(\alpha)$ is also single valued in α. Physically this means that from the point $\boldsymbol{a}$ it must be possible to see the entire boundary of the feasible region (e.g., this condition is satisfied everywhere within a convex probability space). In addition, for this method to be useful the functions $r(\alpha)$ and $\eta[r(\alpha), \alpha]$ must be efficiently evaluatable; the first is primarily a function of the complexity of the constraints, whereas the second is primarily a function of the form of the joint probability density function. Assuming that these conditions are met, Φ_n can then be estimated by generating a sequence of M n-dimensional unit vectors ξ_k with a uniform random distribution over the generalized angle α, evaluating $\eta(\xi_k)$ for each ξ_k and then simply applying the Monte Carlo equations of Appendix A, Eq. (A2), as

$$\hat{\eta} \equiv \frac{1}{M}\sum_{1}^{M}\eta(\xi_k) \qquad \hat{\eta}^2 \equiv \frac{1}{M}\sum_{1}^{M}[\eta(\xi_k)]^2 \tag{8}$$

$$R_\Omega = \Phi_n(\boldsymbol{a}, \Omega) = \int_0^1 \eta(\alpha)\,\mathrm{d}\alpha \approx \hat{\eta} \tag{9}$$

The expected error in R_Ω can be roughly estimated by the one standard deviation error estimate,

$$E^{(R)} = \sqrt{\frac{\hat{\eta}^2 - (\hat{\eta})^2}{M}} \tag{10}$$

Generation of Random Pointing Vectors

For this method to work correctly, it is very important that the M unit vectors ξ_k be distributed uniformly over α, which is not the same thing as M vectors, each of n elements, with each element having a uniform distribution (what is usually meant by the term uniform random vectors). A useful method for generating these random shooting vectors is to apply the results of Appendix B, where it is shown in Eq. (B5) that points generated such that each coordinate has an independent normal Gaussian distribution are themselves distributed uniformly in α, i.e., their distribution is a function of ρ_n only. Thus a unit vector ξ_k can be generated by first generating a vector where each element has a normal Gaussian distribution and then making it a unit vector by dividing each element by the length (Euclidean norm) of the vector. Note that this method applies regardless of the actual joint probability distribution to be integrated; it is merely a convenient way of generating random pointing vectors.

Mixed Probabilistic and Deterministic Parameters

Probability integrals are taken over the entire $\boldsymbol{a}$ subspace, regardless of a parameter's membership in the $\boldsymbol{x}$ or $\boldsymbol{y}$ subsets. In particular, this means that the unit vectors ξ_k should always have zeros in elements corresponding to the deterministic parameters of the $\boldsymbol{b}$ subset (i.e., unit projection onto the $\boldsymbol{a}$ subspace, zero projection onto the $\boldsymbol{b}$ subspace).

Calculating $r(\xi_k)$ for a Particular Constraint

Consider by way of example the problem of calculating the distance r from the point $\boldsymbol{Q}$ to the single constraint $g(\boldsymbol{Q}) = 0$ along the unit vector ξ, where g is (or has been approximated as) a linear function. Then

$$g(\boldsymbol{Q} + r\xi) = g(\boldsymbol{Q}) + r\xi \cdot \nabla \boldsymbol{g} = 0 \tag{11}$$

$$r = -g(\boldsymbol{Q})/[\xi \cdot \nabla \boldsymbol{g}] \tag{12}$$

where the $\cdot$ symbol denotes the standard vector inner product. If r evaluates as negative, it indicates that ξ points away from the constraint, and so r should be taken as $+\infty$ or $\eta(\xi) = 1$.

Similarly, if the constraint $g(\boldsymbol{Q})$ is (or has been approximated as) a quadratic function in $\boldsymbol{Q}$ then the distance r from $\boldsymbol{Q}$ to $g = 0$ along the unit vector ξ is

$$g(\boldsymbol{Q} + r\xi) = g(\boldsymbol{Q}) + r\xi \cdot \nabla \boldsymbol{g} + (r\xi)^T[H](r\xi)$$

$$= C + Br + Ar^2 = 0 \tag{13}$$

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{A} \tag{14}$$

where $[H]$ is the Hessian matrix and

$$A = \xi^T[H]\xi \qquad B = \xi \cdot \nabla \boldsymbol{g} \qquad C = g(\boldsymbol{Q}) \tag{15}$$

and r should be taken as the smallest positive real root, if any (if the constraint g is convex everywhere, and $\boldsymbol{Q}$ is in the feasible region, there can never be more than one positive real root). If there are none it indicates that ξ points away from the constraint, and so r should be taken as $+\infty$ or $\eta(\xi) = 1$.

The Hessian matrix of Eq. (15) is generally expensive to evaluate, and although explicit formulas for determining r can be found for other simple constraint types as well, in many cases r will have to be evaluated numerically. This problem is greatly simplified, however, by the dead band effect discussed later. In that discussion it is shown that explicit roots for r need only be sought within the relatively narrow and well-defined region of the active band, making the problem well bounded.

Estimation of Reliability Derivatives

If the joint probability density function of Eq. (7) is known, it is also possible to efficiently estimate the derivatives of the reliability integrals at the same time as we evaluate the integrals themselves. Direct differentiation of the Monte Carlo equations (8) and (9) with respect to some parameter Q_i yields

$$\frac{\partial}{\partial Q_i}R = \frac{1}{M}\sum_{1}^{M}\frac{\partial}{\partial Q_i}\eta(\xi_k) \tag{16}$$

which is simply a Monte Carlo estimation of the differential change in R due to a differential change in Q_i (i.e., a small shift in the nominal design point). But for any given value of ξ_k the function $\eta(\xi_k)$ is a function only of $r_k = r(\xi_k)$, the distance to the constraint along ξ_k, so that

$$\frac{\partial}{\partial Q_i}\eta(\xi_k) = \frac{\partial}{\partial Q_i}\eta_k(r_k) = \left[\frac{\partial \eta_k(r_k)}{\partial r_k}\right]\left[\frac{\partial r_k}{\partial Q_i}\right] \tag{17}$$

The first term, $[\partial\eta/\partial r]$, is obviously equal to the integrand of Eq. (7), evaluated at $\alpha = r_k$. Note that for a general density function the coefficients of $\eta(r)$ will change for each ξ_k as the components of the unit shooting vector change. Spherically symmetric distributions, such as the Gaussian distribution to be discussed, are an exception to this rule; as for them $\eta(r)$ is the same in all shooting directions.

The second term, $[\partial r/\partial Q_i]$, is essentially a geometric quantity measuring how much r_k must grow or shrink to make $\boldsymbol{Q} + r_k\xi_k$ still just touch the fixed constraint boundary as we move the origin a small distance along the Q_i axis. To first order, this is a function only of the derivatives of the constraint at the point where the shooting vector touches it, as

$$\frac{\partial}{\partial Q_i}r(\xi_k) = -\left[\frac{1}{(\xi_k \cdot \nabla \boldsymbol{g})}\right]\frac{\partial g}{\partial Q_i} \tag{18}$$

Note that there may also be additional terms if r is computed in a normalized space, and if that scaling is itself also a function of Q_i, as in Eq. (29). Note also that although the reliability integral is taken over the $\boldsymbol{a}$ subspace only, its value also depends on the values of the deterministic subset $\boldsymbol{b}$, and so derivatives with respect to elements in the $\boldsymbol{b}$ subset will not necessarily be zero. Finally, the errors in the estimated derivatives can themselves be estimated, by simply applying Eq. (10) to the Monte Carlo evaluation of Eq. (16), i.e., using the mean and mean-square values of each derivative.

Estimation of System Reliability and Its Derivatives

The SMC may be used to estimate the reliability of any region, including either those formed by a single constraint, or the system feasible region bounded by all of the constraints of Eq. (4). Consider now the evaluation of $r^{(S)}$ along the unit vector ξ_k for the net feasible space bounded by the set of J constraints, $g_j(\boldsymbol{Q})$. Then the distance to the system boundary is just the smallest of the $r^{(j)}$ calculated for each individual constraint along that unit vector ξ_k,

$$r^{(S)} = \min_{j=1}^{J} r^{(j)} \tag{19}$$

and $\eta^{(S)}(\xi_k)$ will likewise be the smallest $\eta^{(j)}(\xi_k)$.

In addition, if the critical constraint corresponding to $r^{(S)}$ for this particular ξ_k is labeled as $g^{(S)}$, then it is reasonable to assume

that a differential change in the nominal values Q_i will not change which constraint would be critical for the same ξ_k. For this reason the differential term $\partial\eta/\partial Q_i$ of Eq. (16) for the system reliability will simply be equal to the corresponding term for the critical constraint $g^{(S)}$.

Specialization of the SMC Method to the *n*-Dimensional Gaussian Distribution

Many typical engineering properties exhibit distributions that are Gaussian [Eq. (B2)] or near Gaussian. An n-dimensional Gaussian distribution is one in which each of the n independent probabilistic variables have Gaussian distributions, resulting in a joint probability density function of the form given in Eq. (B5). Some general results for n-dimensional Gaussian distributions are given in Appendix B; the most important of these is that this distribution is spherically symmetric in the normalized probability space. For the SMC method this means that each $\eta(\xi_k)$ is a function only of $r(\xi_k)$, the normalized distance to the constraint boundary along ξ_k. They are not functions of the specific probability density profile along ξ_k, because this is now the same for all ξ_k.

Calculating $\eta(\xi_k)$ for Gaussian Distributions

If each of the parameters in $\boldsymbol{a}$ has a Gaussian distribution, with their nominal values in $\boldsymbol{Q}$ taken as their mean values μ, then if $\boldsymbol{Q}$ is in the feasible region, the origin of the shooting method will be placed at $\boldsymbol{a}$ with each axis scaled by its corresponding standard deviation $\sigma_i(a_i)$. Then the joint probability distribution will be spherically symmetric about $\boldsymbol{a}$, and

$$\eta(\xi_k) = \Psi_n[r(\xi_k)] \tag{20}$$

where r is now the normalized radius to the constraint boundary and Ψ_n is the spherical cumulative probability distribution function [Eq. (B9)].

Calculating $r(\xi_k)$ and Its Derivatives in Scaled Space

In terms of the normalized space, Eq. (12) can be rewritten as

$$r = -\frac{g(\boldsymbol{Q})}{\tilde{\xi}\cdot\tilde{\nabla g}} \tag{21}$$

where $\tilde{\nabla g}$ is the normalized gradient of g,

$$\tilde{\nabla g} = \nabla g \otimes \sigma \Rightarrow \tilde{\nabla g}_i = \nabla g_i \sigma_i \tag{22}$$

and the $\otimes$ operator here denotes a term by term multiplication of one vector by another, to form a new scaled vector. For simple scaling transformations, such as this, it may be more convenient to scale the shooting vector instead:

$$\tilde{\xi} = \xi \otimes \sigma \Rightarrow \tilde{\xi}_i = \xi_i \sigma_i \tag{23}$$

and then use

$$r = -\frac{g(\boldsymbol{Q})}{\tilde{\xi}\cdot\nabla g} \tag{24}$$

Similarly, Eq. (14) can be transformed into the scaled space by simply redefining its coefficients as

$$A = \xi^T[\tilde{H}]\xi = \tilde{\xi}^T[H]\tilde{\xi} \qquad B = \xi\cdot\tilde{\nabla g} = \tilde{\xi}\cdot\nabla g \qquad C = g(\boldsymbol{Q}) \tag{25}$$

where $\tilde{\nabla g}$ and $[\tilde{H}]$ are the normalized gradient and Hessian matrix, respectively.

Calculating Derivatives of *R* for a Gaussian Distribution

Consider Eq. (16), repeated here for convenience:

$$\frac{\partial}{\partial Q_i}R = \frac{1}{M}\sum^{M}\frac{\partial}{\partial Q_i}\eta(\xi_k) \tag{26}$$

For the Gaussian distribution, the nominal values $\boldsymbol{Q}$ represent specifically the mean values μ of the parameters, and it may be necessary (e.g., for sensitivity-based numerical optimization methods) to determine the derivatives of R with respect to some $Q_i = \mu_i$ in either the $\boldsymbol{a}$ or $\boldsymbol{b}$ sets. Differentiation of Eq. (20) yields

$$\frac{\partial}{\partial\mu_i}\eta(\xi_k) = \left[\frac{\partial}{\partial r}\Psi_n(r)\right]\left[\frac{\partial}{\partial\mu_i}r(\xi_k)\right] \tag{27}$$

The first term can be found by differentiating Eq. (B8), which simply yields the value of the integrand at $\rho_n = r$, as

$$\frac{\partial}{\partial r}\Psi_n(r) = \left(\frac{\overline{2}}{\Gamma(n/2)}\right)(e^{-\frac{1}{2}r^2})\left(\frac{r^2}{2}\right)^{(n-1)/2} \tag{28}$$

As already discussed, the second term, $[\partial r/\partial\mu_i]$, can be approximated to first order by considering the constraint linearized at the point where the shooting vector touches the boundary and directly differentiating Eq. (21) as

$$\frac{\partial}{\partial\mu_i}r(\xi) = -\left[\frac{1}{(\xi\cdot\tilde{\nabla g})}\right]\left[\frac{\partial g}{\partial\mu_i} + (r)\left(\xi\cdot\frac{\partial}{\partial\mu_i}\tilde{\nabla g}\right)\right] \tag{29}$$

where, from Eq. (22),

$$\frac{\partial}{\partial\mu_i}\tilde{\nabla g} = \left(\frac{\partial}{\partial\mu_i}\nabla g\otimes\sigma\right) + \left(\nabla g\otimes\frac{\partial}{\partial\mu_i}\sigma\right) \approx \nabla g\otimes\frac{\partial}{\partial\mu_i}\sigma + \mathcal{O}(\epsilon^2) \tag{30}$$

The first term in the last brackets of Eq. (29) represents the change in r due to a simple translation of the base point of the shooting vector, causing its tip to either grow or shrink as required to still just touch the boundary. The second term represents the second-order effect of a nonlinear constraint, which has been neglected, and the effect on r caused by the change in scaling σ, when $\sigma = \boldsymbol{f}(\mu)$. By our assumption of independence among the parameters, the only possible nonzero term of $\partial\sigma/\partial\mu_i$ is $\partial\sigma_i/\partial\mu_i$.

Again, because this differential is essentially a local effect at the end of the shooting vector, Eq. (29) can also be applied to nonlinear constraints, where the indicated derivatives of g are taken at the point of intersection of the shooting vector and the constraint, while the derivatives $(\partial\sigma/\partial\mu_i)$ are taken at the mean point $\mu = \boldsymbol{Q}$.

Applying the SMC Method to Non-Gaussian Distributions

Depending on the specific forms of the various probability density functions assumed for the elements of the $\boldsymbol{a}$ set, it may not always be possible to generate scalings that make the joint probability density of Eq. (5) spherically symmetric. In these cases $\eta(\xi_k)$ will remain a function of the shooting direction ξ_k, as well as the distance $r(\xi_k)$, to the constraint boundary. Because the integrand of Eq. (7) is a simple product of independent terms, it may still be possible to derive efficient analytic forms for this evaluation.

In a worst-case scenario, where complex forms of the probability density function preclude finding an analytical form of $\eta(\xi_k)$, it may be necessary to numerically integrate from the origin to the boundary along each ξ_k. This process is aided by the fact that the integral must always be monotonically increasing from 0 to a maximum less than or equal to 1, and some form of the dead band effect (to be discussed) will generally limit the maximum distance that must be considered. This paper presents a comparison of convergence rates for SMC vs CMC, and there it is shown that in most cases the SMC method will require many fewer shooting vectors than an equivalent CMC analysis. This will usually make the SMC method more attractive even where numerical integration is required and even more so when derivatives are also required {note that the term $[\partial\eta/\partial r]$ of Eq. (17) will automatically be calculated as the terminal value of the integrand}.

As a final comment, note that in most cases it will be preferable to wrap all of the probability coefficients that are functions of the design variables into the definition of r, e.g., as scaling factors, leaving $\eta(r)$ as a true function of the scaled r only. This is simply because it will generally be easier to include their effects on Eq. (17) in the differentiation of $r(\boldsymbol{Q})$, rather than the differentiation of $\eta(r)$.

Dead Band Effect and Modeling Accuracy

Figure 2 plots the total probability contained within an n-dimensional hypersphere of normalized radius ρ_n. The hypersphere is centered at the mean of a normalized Gaussian distribution [Eq. (B5)], with the corresponding spherical cumulative probability function derived in Eq. (B9). As the dimensionality n of the probability space increases, some distinctive characteristics can be seen.

1) Each curve is preceded by a region $0 \leq \rho_n \leq \rho_{\min}$, the inner dead band, in which the enclosed probability remains essentially zero. In terms of the CMC method, this represents a region in which very few test points will fall, despite this region having the highest probability density, because of the volumetric growth characteristics of hypergeometric spaces. The key point here is that probability calculations are relatively insensitive to the accuracy of constraint boundaries that lie within the dead band, as long as they are correctly identified as being somewhere within it. This observation becomes explicit in the SMC method, where from Eq. (20) one can see that for all $r(\xi_k) \leq \rho_{\min}$ the probability contribution $\eta(r)$ will be close to zero, as will the derivative contribution of Eq. (27) (because $\partial\Psi_n/\partial r$ also approaches zero).

2) Similarly, each curve is followed by a region $\rho_{\max} \leq \rho_n \leq \infty$ the outer dead band, which in the CMC method will also receive very few test points, in this case because the falling probability density is stronger than the volumetric effect (or, equivalently, because the enclosed probability is already approaching 1.0). Again one can see that probability calculations are relatively insensitive to the accuracy of constraint boundaries that lie within this dead band; for the SMC they will all have $\eta(r) \approx 1$ and $\partial\Psi_n/\partial r$ near zero.

3) The region $\rho_{\min} \leq \rho_n \leq \rho_{\max}$ is, by extension, the active band, and it is here that it is most important to accurately model the constraint boundaries.

The values of $\rho_{\min}$ and $\rho_{\max}$ can be defined more explicitly by selecting some small parameter $\varepsilon \ll 1$, with

$$\rho_{\min} = \Psi_n^{-1}(\varepsilon) \qquad \rho_{\max} = \Psi_n^{-1}(1 - \varepsilon) \tag{31}$$

where ε represents the total cumulative probability that could be included within the inner or outer dead band and, hence, the maximum possible error could be introduced by not locating such constraints beyond their mere existence somewhere in the band. Table 1 shows some representative values for different values of ε and n; the most interesting thing about these results is how little $\rho_{\min}$ and $\rho_{\max}$ change as ε changes by four orders of magnitude.

Note that this requirement for accuracy throughout the active band is in marked contrast to conventional deterministic optimization, where the only theoretical requirement of an approximate model (or the terminus of a sequence of approximate models) is that it be exact in its values and derivatives of the binding constraints at the singular point of the optimum itself (e.g., Ref. 1). Probability calculations are inherently volumetric in nature, and the critical modeling region is not at the singular point of the mean values chosen, but rather everywhere in the hyperspherical active band. This places much greater demands on any approximate model used for evaluating reliabilities, and these demands increase with both n, the number of probabilistic parameters used, and the standard deviations σ of these parameters, both of which increase the length of $\rho_{\max}$ in nonscaled (model) space. On the other hand, the start of the outer dead band does at least limit the range over which modeling accuracy is important for reliability calculations, and Eq. (30) can be used to determine the maximum possible error that could be incurred by simply ignoring altogether any constraints that might fall outside of this region.

Table 1 Values of $\rho_{\min}$ and $\rho_{\max}$ for various values of ε and n

ε, %	$n = 4$		$n = 10$		$n = 20$	
	$\rho_{\min}$	$\rho_{\max}$	$\rho_{\min}$	$\rho_{\max}$	$\rho_{\min}$	$\rho_{\max}$
1	0.545	3.644	1.599	4.818	2.874	6.129
0.01	0.169	4.849	0.943	5.963	2.096	7.238
0.0001	0.053	5.777	0.581	6.846	1.598	8.088

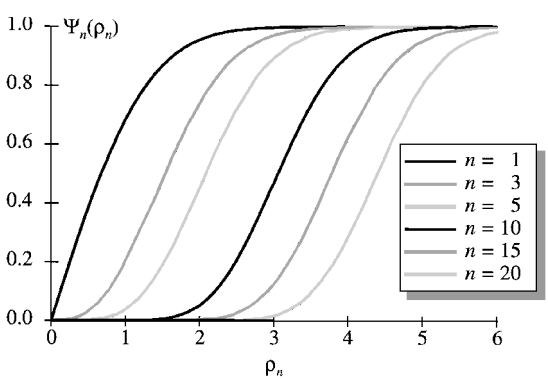

Fig. 2 Cumulative probability contained within an n-dimensional hypersphere of radius ρ_n.

Dead Band Effect for Non-Gaussian Distributions

Although this derivation of the dead band effect is specifically for a Gaussian distribution, the root causes of the inner and outer dead bands will cause similar effects in any nonsingular probability distributions. The inner dead band is basically due to the volumetric nature of hyperspace [Eq. (B7)]; differential volumes fall to zero as $\rho_n \to 0$ and ever more sharply as the dimensionality n increases, and so for a finite probability density this contribution to the cumulative probability must also vanish. Similarly, the outer dead band is simply caused by the requirement that as ρ_n becomes large the contribution to the cumulative probability for any nonsingular distribution must again vanish, in order to keep the total cumulative probability less than or equal to one (this means that the density must not only fall off, it must fall off faster than the differential volume elements are increasing).

Implications of the Dead Band Effect for SMC

Despite the demands that the dead band effect places on reliability models in general, it is overall probably of net benefit to the SMC method. For the Gaussian distribution in particular, it was shown that the key problem for each shooting vector is solving for the distance r to the nearest constraint. If the constraint formulations are too complex to derive an explicit method of calculating r, or the constraints are computed externally, it may be necessary to evaluate r by numeric root finding along ξ_k. The dead band effect, however, effectively brackets the region in which roots must be sought: 1) $0 \leq r \leq \rho_{\min}$, if a root $g(r\xi_k) = 0$ exists anywhere in the inner dead band, then let $r \to 0$, $\eta[r(\xi_k)] = 0$, $\partial\eta/\partial r = 0$; 2) $\rho_{\min} \leq r \leq \rho_{\max}$, else if a root $g(r\xi_k) = 0$ exists anywhere in the active band, then actually seek the position of that root and solve for $\eta[r(\xi_k)]$ and $\partial\eta/\partial r$ as discussed; and 3) $\rho_{\max} \leq r$, else the root, if any, is in the outer dead band, let $r \to \infty$, $\eta[r(\xi_k)] = 1$, $\partial\eta/\partial r = 0$.

Comparison of Convergence Rates for SMC vs CMC

As discussed in Appendix A, the accuracy of a reliability estimate for $R_S > 0.5$ can best be expressed in terms of its expected error $E^{(R)}$ relative to the probability of failure, $F = 1 - R$:

$$\varepsilon^{(F)} = E^{(R)}/F = E^{(R)}/(1 - R) \tag{32}$$

Comparing Eq. (10) for SMC and Eq. (A11) for CMC, it is clear that $E^{(R)}$ in both cases is inversely proportional to $\sqrt{M}$, where M is equal to the number SMC shooting vectors or CMC design point analyses required. Inverting this relationship, one can define the parameter K as

$$M = \frac{1}{\left[\varepsilon^{(F)}\right]^2} K \tag{33}$$

where K is now a measure of the expense of using one method vs the other. For CMC K is a function of R only, without regard to the dimensionality or shape of the joint probability density or the feasible region. From Eq. (A15),

$$K_{\text{CMC}} = R/(1 - R) \tag{34}$$

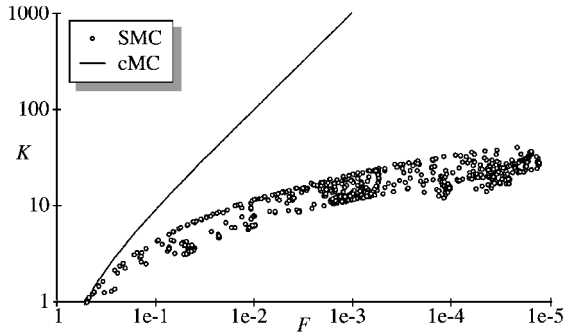

Fig. 3 K_{SMC} values based on SMC evaluations of R_S for sample problem 1.

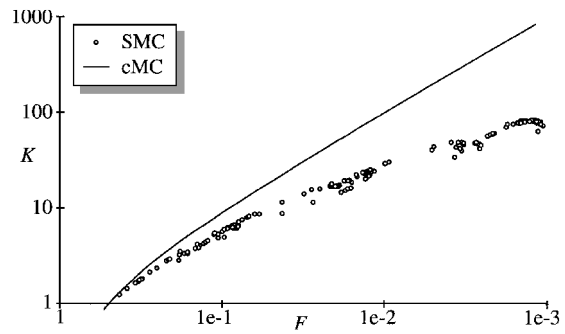

Fig. 4 K_{SMC} values based on SMC evaluations of R_S for sample problem 2.

From Eq. (10), the value of K for SMC will be equal to

$$K_{SMC} = \frac{\hat{\eta}^2 - (\hat{\eta})^2}{(1 - R)^2} \tag{35}$$

which depends not only on R but also on the distribution of the η_k generated, which will be problem dependent. The η_k, however, are bounded such that as R approaches 1 so will the η_k, making K_{SMC} indeterminant as $R \longrightarrow 1$, rather than strictly infinite as is the case for K_{CMC}.

The actual K_{SMC} values found for two sample problem spaces are shown in Figs. 3 and 4, along with their equivalent K_{CMC} curves from Eq. (34). These sample problems are discussed in detail in Refs. 2 and 3, where they appear as problems 2 and 4. In summary, problem 1 is an actively controlled cantilever beam with four probabilistic parameters ($n = 4$) and five steady-state dynamic constraints. For these runs, $\varepsilon^{(F)} \approx 1\%$, M(min, mean, max) = (1.0e4, 2.5e5, 1.3e6). Problem 2 is the standard 10 bar truss problem, in this case under harmonic load. This problem has 10 probabilistic parameters ($n = 10$) and 18 steady-state dynamic constraints. For these runs, $\varepsilon^{(F)} \approx 0.5\%$, M(min, mean, max) = (1.2e5, 3.9e5, 8.5e5).

An example of the ability of SMC to estimate derivatives of the reliabilities is presented in Refs. 2 and 3.

Conclusions

The SMC method presented here offers two important advantages over the CMC method: it estimates the reliabilities more efficiently, and it can provide high-quality estimates of the derivatives of R_S concurrently with the estimation of R_S itself. SMC offers significantly greater efficiency in estimating the system reliability, particularly for systems of high reliability where $R_S \longrightarrow 1$. As demonstrated in Figs. 3 and 4, the number of SMC vectors required may be as much as several orders of magnitude fewer than the equivalent number of CMC design evaluations. This gain in efficiency is offset somewhat by the necessity of finding the distance along each SMC shooting vector to the nearest constraint boundary, which, for the general case, will be done numerically. This, however, is merely a standard one-dimensional root finding problem, with the roots of interest well bounded by the dead band effect. Thus if the constraints are reasonably smooth, each SMC vector will require one design evaluation (at ρ_{min}) if the boundary is in the inner dead band, an additional evaluation (at ρ_{max}) if the boundary is in the outer dead band, and, typically, an additional 3–5+ evaluations if the boundary is in the active band.

Based on results to date, this indicates that SMC will be more efficient than CMC, even in terms of total number of design evaluations, once the failure probability drops below about 1% (note that when the boundary distances can be found explicitly SMC will always be more efficient than CMC).

In addition, the ability of SMC to estimate derivatives of the reliabilities is crucial to allowing its use in gradient-based optimization schemes. Examples of this are presented in Refs. 2 and 3.

Appendix A: Summary of the CMC Approach

The CMC approach to evaluating the integral of a general function proceeds by evaluating the function at each of a set of randomly selected test points uniformly distributed over the region of the integral, or some convenient enveloping region.[4] This form is generally not useful for the integration of probability density functions because their domain is typically infinite; thus, an alternate method is used, which generates test points with a distribution equal to the actual probability density of the probability function to be integrated and then simply counts the number of test points that fall within the region of the integral. This method may be called a direct Monte Carlo simulation, as opposed to the usual meaning of the term Monte Carlo integration.

Monte Carlo Integration

Consider the computation of the integral of the function $f(\boldsymbol{x})$ over some region Ω:

$$F = \int_{\Omega} f(\boldsymbol{x})\, d\Omega \tag{A1}$$

This integral may be approximated by evaluating $f(\boldsymbol{x}_i)$ at a set of M points $\boldsymbol{x}_i$ distributed uniformly over the region Ω, and using the Monte Carlo equations

$$\hat{f} \equiv \frac{1}{M}\sum_{i}^{M} f(\boldsymbol{x}_i) \qquad \hat{f}^2 \equiv \frac{1}{M}\sum_{i}^{M} [f(\boldsymbol{x}_i)]^2 \tag{A2}$$

$$F \approx V\hat{f} \pm E^{(F)} \qquad E^{(F)} = V\sqrt{\frac{\hat{f}^2 - (\hat{f})^2}{M}} \tag{A3}$$

where V is the generalized volume of the region Ω, i.e.,

$$V = \int_{\Omega} d\Omega$$

The one standard deviation error estimate $E^{(F)}$ is based on the assumption that f has a Gaussian distribution but seems to work well as a rough measure of error for most practical problems.

It is instructive to rearrange the terms of Eqs. (A2) and (A3) as

$$F = \int_{\Omega} f(\boldsymbol{x})\, d\Omega \approx \sum_{i}^{M} f(\boldsymbol{x}_i)\left[\frac{V}{M}\right] = \sum_{i}^{M} [f(\boldsymbol{x}_i)V]\frac{1}{M} \tag{A4}$$

The first of these forms makes explicit the relationship between $d\Omega$ and V/M; as $M \longrightarrow \infty$ $V/M \longrightarrow d\Omega$. The second form is of computational interest, as the term $[f(\boldsymbol{x}_i)V]$ is simply equal to the value the integral F would have if $f(\boldsymbol{x})$ were equal to a constant $f(\boldsymbol{x}_i)$ everywhere in Ω. If, as frequently happens, the region Ω is not convenient to work with (i.e., it is difficult to find its volume and/or generate points uniformly over the region) one can simply use a new region Ω', which completely encompasses the original region, and a new function $f'(\boldsymbol{x})$, such that

$$f'(\boldsymbol{x}) \equiv \begin{cases} f(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega \\ 0 & \text{if } \boldsymbol{x} \notin \Omega \end{cases} \tag{A5}$$

Monte Carlo Simulation

Monte Carlo simulation is used to directly estimate reliabilities by generating a set of M test points (instances) $\boldsymbol{x}_i$ with a distribution equal to that of the general population of $\boldsymbol{x}$. Using the direct simulation method, if the reliability of a system (i.e., the probability that any given test point falls within the region of the integral) is R, then the number of test points that fall within that region will

be R times M. This can be thought of as taking the integral of the joint probability density function and transforming it into the domain of the probability density function itself and then using an encompassing integrand volume Ω_I, which covers all space, so that $V = \int \phi_n = 1$. To make this correlation between Monte Carlo integration and simulation a bit more explicit, consider the following integral of a one-dimensional probability density function:

$$R = \int_{\Omega} \phi(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx \qquad f(x) \equiv \begin{cases} \phi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases} \tag{A6}$$

Apply the substitution

$$ds \equiv \phi(x)\,dx \Longrightarrow s = \Phi(x) \qquad x = \Phi^{-1}(s) \tag{A7}$$

Then

$$R = \int_{\Omega} ds = \int_{0}^{1} f'(s)\,ds \qquad f'(s) \equiv \begin{cases} 1 & \text{if } x = \Phi^{-1}(s) \in \Omega \\ 0 & \text{if } x = \Phi^{-1}(s) \notin \Omega \end{cases} \tag{A8}$$

Note, in particular, that the transformation $x = \Phi^{-1}(s)$ converts a uniform distribution in s, $0 \leq s \leq 1$, to the original distribution $\phi(x)$ in x, $-\infty \leq x \leq \infty$; this means that a Monte Carlo integration in s will generate a set of test points that simulate drawing a random set of M instances from the original population of x.

Because each of the n probabilistic parameters is considered to be independent, this result can be immediately extended to the n-dimensional case, with each x_i generated independently according to its own distribution as $x_i = \Phi_i^{-1}(s_i)$. The transformed integrand becomes

$$f'(\mathbf{x}) \equiv \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega \\ 0 & \text{if } \mathbf{x} \notin \Omega \end{cases} \tag{A9}$$

and the equivalent Monte Carlo equations become

$$\hat{f} = (1/M)(RM)(1) = R \qquad \hat{f}^2 = (1/M)(RM)(1^2) = R \tag{A10}$$

$$R \approx R \pm E^{(R)} \qquad E^{(R)} = \sqrt{\frac{R - (R)^2}{M}} \tag{A11}$$

or

$$E^{(R)} = (1/\sqrt{M})\sqrt{R(1-R)} \tag{A12}$$

with the fractional error bound given by

$$\varepsilon^{(R)} = E^{(R)} / R = (1/\sqrt{M})\sqrt{(1-R)/R} \tag{A13}$$

At first glance this looks pretty good, because designs are generally intended to operate in a region of high reliability, and as $R \longrightarrow 1$ we see that $E^{(R)}$ and $\varepsilon^{(R)} \longrightarrow 0$. Unfortunately, we must first address the question of what constitutes an acceptably small error bound. If the value of R is 0.5, then an $E^{(R)}$ of 0.005 ($\varepsilon^{(R)} = 1\%$) is probably quite reasonable; but, paradoxically, if R is 0.999 an $E^{(R)}$ of 0.005 ($\varepsilon^{(R)} \approx \frac{1}{2}\%$) is probably not acceptable, even though it represents a smaller fractional error. This is because for high values of R what is actually important is the accuracy with which the probability of failure, $F = 1 - R$, has been evaluated. Substitution into Eq. (A12) yields

$$\varepsilon^{(F)} = E^{(R)} / F = (1/\sqrt{M})\sqrt{R/(1-R)} \tag{A14}$$

which goes to infinity as $R \longrightarrow 1$. An equivalent way of stating this is that the number of Monte Carlo evaluations M required to maintain a constant error will increase to infinity as $R \longrightarrow 1$,

$$M = \frac{1}{\left[\varepsilon^{(F)}\right]^2} \frac{R}{1-R} \tag{A15}$$

Appendix B: Integrals of the Gaussian (Normal) Distribution Function

The first and second statistical moments (mean and variance) of a population can be estimated from a set of N discrete instances of that population, or from a known continuous probability density $\Phi(x)$, as

$$\begin{array}{lll} & \text{Discrete} & \text{Continuous} \\ \mu = \hat{x} = & \dfrac{1}{N}\displaystyle\sum_{i}^{N} x_i & = \displaystyle\int_{-\infty}^{\infty} \phi(x)\,x\,dx \\ \sigma^2 = \hat{x}^2 = & \dfrac{1}{N-1}\displaystyle\sum_{i}^{N} (x_i - \mu)^2 & = \displaystyle\int_{-\infty}^{\infty} \phi(x)(x-\mu)^2\,dx \end{array} \tag{B1}$$

It is very common to find that the statistical distribution of a parameter will follow, at least approximately, a Gaussian distribution, which is also commonly known as the normal distribution. This distribution can be defined completely in terms of its mean value μ, which is equal to its first moment, and its standard deviation σ, which is equal to the square root of its second moment (variance). The general Gaussian distribution function $\phi(v)$ for a single variable v with a mean value of μ and a standard deviation of σ is given by

$$\phi(v) = \phi(\mu, \sigma^2) = \left[1/(\sigma\sqrt{2\pi})\right] \exp\left[-\tfrac{1}{2}(v-\mu)^2/\sigma^2\right] \tag{B2}$$

Note that $\phi(v)$ and $\phi(\mu, \sigma^2)$ are not two different distribution functions, but merely two ways of expressing the same function, i.e., the distribution of v is defined in terms of its mean μ and variance σ^2. The probability that the actual value of (some specific instance of) v is less than or equal to a given value z is then

$$P(v \leq z) = \Phi(z) = \int_{-\infty}^{z} \phi(v)\,dv \tag{B3}$$

By translating the origin of v to its mean value and scaling by its standard deviation, this can be transformed into the normal form:

$$\begin{gathered} \delta v \equiv \frac{v-\mu}{\sigma} \qquad d\delta v = \frac{dv}{\sigma} \\ \Phi\left(\delta z \equiv \frac{z-\mu}{\sigma}\right) = \int_{-\infty}^{\delta z} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\delta v^2\right] d\delta v \end{gathered} \tag{B4}$$

Note that by definition normalized parameters have their mean at $\delta v = 0$ and standard deviation $\sigma = 1$.

Consider now a space of n independent parameters a_i in normalized space, and let δa_i represent the (normalized) variance of some actual instance of a_i from the mean of its population. Because they are independent, the joint probability density function $\phi_n(\delta \mathbf{a})$ is just the product of their individual probability density functions:

$$\begin{aligned} \phi_n(\delta \mathbf{a}) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\delta a_i^2\right) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i}^{n} \delta a_i^2\right) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\rho_n^2\right) \end{aligned} \tag{B5}$$

and so the total probability that the actual value of $\delta \mathbf{a}$ will lie within some region Ω is just

$$\Phi_n(\delta \mathbf{a} \in \Omega) = \int_{\Omega} \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\rho_n^2\right) d\Omega \tag{B6}$$

Note that the resultant probability density function is a function of ρ_n only, the normalized distance from the origin (mean) to the actual value of $\mathbf{a}$, and so is spherically symmetric in normalized space.

For the probability in a hypersphere, consider the case of a normalized n-dimensional probability space, which contains a hypersphere centered at the origin, with a radius of ρ_n. The generalized differential volume (i.e., generalized surface area) of a hypersphere is shown in Ref. 2 to be

$$S_n r^{n-1} = 2\frac{\pi^{\frac{1}{2}n}}{\Gamma(n/2)} r^{n-1} \tag{B7}$$

where $\Gamma(\ )$ is the standard gamma function. Thus, the probability that the normalized radius r from the origin (mean) to the actual value of $\boldsymbol{a}$ is less than or equal to ρ_n is

$$\Phi_n(\rho_n) = \int_0^{\rho_n} \frac{2\pi^{\frac{1}{2}n}}{\Gamma(n/2)} r^{n-1} \left(\frac{1}{\sqrt{2\pi}} \right) e^{-\frac{1}{2}r^2}\, \mathrm{d}r$$

$$= \int_0^{\rho_n} \left(\frac{\sqrt{2}}{\Gamma(n/2)} \right) \left(e^{-\frac{1}{2}r^2} \right) \left(\frac{r^2}{2} \right)^{(n-1)/2} \mathrm{d}r \qquad \text{(B8)}$$

This hyperspherical cumulative normal distribution function is plotted in Fig. 2. Because it is of particular interest for the SMC method, it will be denoted in this text as $\Psi_n(\rho_n)$. Using the substitution $u = \frac{1}{2}r^2$, $\mathrm{d}u = r\,\mathrm{d}r$, this function can be rewritten in a more standard form as

$$\Psi_n(\rho_n) = \frac{1}{\Gamma(n/2)} \int_0^{\frac{1}{2}\rho_n^2} e^{-u} u^{\frac{1}{2}n-1}\, \mathrm{d}u$$

$$= \wp\left[\frac{1}{2}n, \frac{1}{2}\rho_n^2 \right] = \chi^2(n, \rho_n^2) \qquad \text{(B9)}$$

where $\wp$ is the incomplete gamma function and χ^2 is the chi-squared cumulative distribution function.

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